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***Proper Integral***: The definite integral  is called a proper integral if

(i). the interval of integration  is finite or bounded

(ii). the integrand  is bounded on .

***Improper Integral***: The definite integral  is called an improper integral if either or both the above conditions are not satisfied. Thus  is an improper integral if either the interval of integration  is not finite and the integrand  is not bounded on or neither the interval  is finite nor the integrand is bounded over it.

The improper integrals are of three kinds:

(i). improper integral of first kind.

(ii). improper integral of second kind and

(iii). improper integral of third kind.

***Improper integral of first kind:***In the definite integral , if either  or or both  and  are infinite, so that the interval of integration is unbounded but the integrand is bounded, then the definite integral  is called an improper integral of first kind.

**Example:** , ,  are improper integrals of first kind.

***Improper integral of second kind:***In the definite integral , if both  and  are finite, so that the interval of integration is finite but the integrand  has one or more points of infinite discontinuity, i.e. the integrand  is not bounded on , then the definite integral  is called an improper integral of second kind.

**Example:** ,  are improper integrals of second kind.

***Improper integral of third kind:***In the definite integral , if either  or or both  and  are infinite, so that the interval of integration is unbounded and the integrand is also unbounded, then the definite integral  is called an improper integral of third kind.

**Example:**  is an improper integral of third kind.

***Convergence and divergence of improper integral:*** Determine whether the limit exists. If the limit exists or is finite, then the improper integral is said to be convergent and take the limit for its value. If the limit does not exist or is infinite, then the improper integral is said to be divergent.

***Improper integral of the first kind as the limit of a proper integral:***When the improper integral is of the first kind, either  or or both  and  are infinite but the integrand is bounded.

(i)..

The improper integral  is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is  or .

(ii)..

The improper integral  is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is  or .

(iii).

where  is any real number. The improper integral  is said to be convergent if both the limit on the R.H.S. exists finitely and independent of each other, otherwise the integral is said to be divergent.

**Note:** If the integral is neither convergent nor divergent, then it is said to be oscillating.

**Problem-01:** Test for convergence of .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have

















.

The given integral is convergent and its value is .

**Problem-02:** Test for convergence of .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have



Putting 

when  then 

when  then .

Now from (1), we have













.

The given integral is convergent and its value is .

**Problem-03:** Test for convergence of .

**Solution:** Let 



Now by the definition, we have



Putting 

when  then 

when  then .

Now from (2), we have















Also by the definition, we have



Putting 

when  then 

when  then .

Now from (3), we have















So (1) becomes,

.

This is finite. So the given integral is convergent and its value is 0.

**Problem-04:** Test for convergence of .

**Solution:** Let 

Now by the definition, we have













.

This is infinite. So the given integral is divergent.

**Problem-05:** Test for convergence of .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have



Putting 

when  then 

when  then .

Now from (1), we have



















This is infinite. The given integral is divergent.

**Problem-06:** Evaluate .

**Solution:** Let 





Now,



Putting 

We have

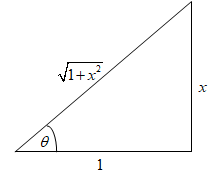






















From (1), we get















 By L. Hospital’s Rule



.

This is finite. So the given integral is convergent and its value is .

**Problem-07:** Show that .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have



Now,



Putting 

We have



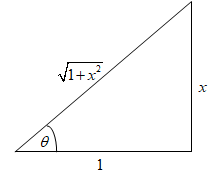






















From (1), we get







 By L. Hospital’s Rule





. **(Showed)**

**Problem-08:** Evaluate .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have























.

**Problem-09:** Evaluate .

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have















.

**Problem-10:** Show that  converges if , it diverges otherwise.

**Solution:** Let 

Since the upper limit of the given integral is , so by the definition we have



when , then



i.e. when , then the given integral is divergent.

when , then

.

Here the sign of  is important. When  that is  then  is in the numerator.

Therefore



Thus the given integral diverges.

When  that is  then  is in the denominator.

Therefore



Thus the given integral converges.

Hence, the given integral converges if  and it diverges otherwise. **(Showed)**

***Improper integral of the second kind as the limit of a proper integral:***When the improper integral is of the second kind, both  and  are infinite but the integrand has one or more points of infinite discontinuity.

(i).If  becomes infinite at only, we define .

The improper integral  is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is  or .

(ii). If  becomes infinite at only, we define .

The improper integral  is said to be convergent if the limit on the R.H.S. exists finitely and the integral is said to be divergent if the limit is  or .

(iii).If  becomes infinite at only where , we define



The improper integral  is said to be convergent if both the limit on the R.H.S. exists finitely and independent of each other, otherwise the integral is said to be divergent.

**Problem-11:** Test for convergence of .

**Solution:** Let 

Since 0 is the only point of infinite discontinuity of the integrand  on , so by the definition we have



Putting  

when  then 

when  then 

Now from (1), we have











.

The given integral is convergent and its value is .

**Problem-12:** Test for convergence of .

**Solution:** Let 

Since 0 and 1 are the points of infinite discontinuity of the integrand  on , so by the definition we have

























.

The given integral is convergent and its value is .

**Problem-13:** Test for convergence of .

**Solution:** Let 

Since 1 is the only point of infinite discontinuity of the integrand  on , so by the definition we have













.

The given integral is divergent.

**Problem-14:** Test for convergence of .

**Solution:** Let 

where 

Now 

So this is an odd function.



Which is finite. So the given integral is convergent.

***Comparison Test:*** Sometimes an improper integral is too difficult to evaluate. One technique is to compare it with a know integral. The test below shows us how to do this.

Let  and  be functions which are continuous on the interval . Suppose that  for all  in .

1. If  is convergent then  is also convergent.
2. If  is divergent then  is also divergent.

***Dirichlet’s Test:*** If  is bounded and monotonic decreasing in the interval ,  and  (a finite number) for finite values of *x* , then is convergent.

**Problem-15:** Test for convergence of .

**Solution:** Let 

Now 

Again 

which is finite. So  is convergent.

So by comparison test, we have  is convergent.

**Problem-16:** Test for convergence of .

**Solution:** Let 



Since  is continuous in  and , so  is a proper integral.

We know that the proper integral being always convergent. So  is convergent.

Now we need only to test the convergence of .

Now by applying Dirichlet’s Test we will examine the convergence of .

Let  and 

 and , so  is monotonic decreasing for all .

Now





Therefore, for all finite value *x* ,  is bounded.

Thus, by Dirichlet’s Test we have , i.e.  is convergent.

So that  is convergent.

**Problem-17:** Test for convergence of .

**Solution:** Let 

Now by applying Dirichlet’s Test we will examine the convergence of .

Let  and 

 and , so  is monotonic decreasing for all .

Now





Therefore, for all finite value *x* ,  is bounded.

Thus, by Dirichlet’s Test we have , i.e.  is convergent.

* If the integrand is a function of one or more parameters in addition to the variable of

integration, then the integral between the limits which may be constants or functions of the parameters is a function of these parameters.



Thus, in general , , where *a*, *b* may be constants or functions of parameters. Sometimes  is such that the evaluation of the integral is very complicated for impossible. However the integral with integrand  may be easily evaluated. Hence we discuss here how to differentiate the integral (i) w.r.to the parameter .

***Leibnitz’s rule for differentiation under the integral sign:*** If  and  are continuous functions of  and for , , *a*, *b* being independent of , then

.

***Working rules for evaluate a given integral :***

1. Let 
2. Differentiate both sides w.r.to  using Leibnitz’s Rule.

Then .

1. Evaluate the integral on R.H.S.
2. Integrate both sides w.r.to , adding the constant of integration on R.H.S.
3. Evaluate the constant of integration by giving suitable value to the parameter .

**Problem-18:** Prove that .

**Solution:** Let 

Differentiating (1) w. r. to *a* under the integral sign, we have











Now integrating (2) w. r. to *a* we get





where *c* is a constant of integration.

Putting  in (3), we get









Using the value *c* in (3), we get

 **(Proved).**

**Problem-19:** Prove that  and hence show that .

**Solution:** Let 

Differentiating (1) w. r. to *b* under the integral sign, we have













Now integrating (2) w. r. to *b* we get







where *c* is a constant of integration.

Putting  in (3), we get







Using the value *c* in (3), we get

 **(Proved).**

**2nd part:** Putting , we get



 **(Proved).**

**Assignment:**

**Problem-01:** Test for convergence of .

**Problem-02:** Test for convergence of .

**Problem-03:** Test for convergence of .

**Problem-04:** Test for convergence of .

**Problem-05:** Evaluate .

**Problem-06:** Evaluate .

**Problem-07:** Test for convergence of .

**Problem-08:** Test for convergence of .

**Problem-09:** Evaluate .

**Problem-10:** Evaluate .

**Problem-11:** Evaluate .

**Problem-12:** Evaluate .